Indecomposability of free nonsingular actions by nonamenable groups in $\mathcal{QH}_{\text{reg}}$

After C. Houdayer and S. Vaes

Brent Nelson

UCLA

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Definition 1.1

Recall that a countably infinite group Γ belongs to the class $\mathcal{QH}_{\text{reg}}$ if it admits

(i) $\pi\colon \Gamma \to \mathcal{U}(H)$ a unitary representation which is weakly contained in the left regular representation;

(ii) $c\colon \Gamma \to H$ a proper map (i.e. $|\{g \in \Gamma\colon \|c(g)\| \leq R\}| < \infty$ for every $R > 0$) satisfying

$$\sup_{x\in\Gamma} \|c(gxh) - \pi_g(c(x))\| < \infty \qquad \forall g, h \in \Gamma.$$

Definition 1.2

- An action $\Gamma \curvearrowright (X, \mu)$ is *non-singular* if $\mu(g \cdot A) = 0 \Leftrightarrow \mu(A) = 0$ for every $g \in \Gamma$ and $A \subset X$.
- Let $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$ be the equivalence relation generated by such a non-singular action. We say it is *recurrent* if for every Borel subset $W \subset X$ with $\mu(W) > 0$, and for μ-almost every $x \in W$, the intersection $W \cap \{y : (x, y) \in \mathcal{R}\}$ is infinite. Equivalently, for μ-almost every $x \in W$, the orbit $\Gamma \cdot x$ returns to W infinitely often. This is also equivalent to saying $L^\infty(X) \rtimes \Gamma$ has no type I direct summand.
- We say $\mathcal{R}$ is *decomposable* if $(X, \mu) = (X_1, \mu_1) \times (X_2, \mu_2)$ and there are recurrent non-singular equivalence relations $\mathcal{S}_i$ on (X_i, μ_i) such that $(x, y) \in \mathcal{R}$ iff $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$ with $(x_i, y_i) \in \mathcal{S}_i$ for $i = 1, 2$, i.e. $\mathcal{R} = \mathcal{S}_1 \times \mathcal{S}_2$.

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- $\eta_\varphi(x) \mapsto \eta_\varphi(x^*)$ for $x \in M$ extends to a densely defined unbounded operator S on $\mathcal{H}_\varphi$.
- Letting $S = J\Delta_\varphi^{\frac{1}{2}}$ be the polar decomposition, we define $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$, the *modular automorphism group of* φ, by $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$.

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- Define a representation π_σ of M on $L^2(\mathbb{R}, \mathcal{H}_\varphi)$, the space of square integrable $\mathcal{H}_\varphi$-valued functions by

$$(\pi_\sigma(x)\xi)(s) = \pi_\varphi\left(\sigma_{-s}^\varphi(x)\right)\xi(s),$$

and let $\mathbb{R}$ act via translation: $(\lambda(t)\xi)(s) = \xi(s-t)$.

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- The *crossed product* $M \rtimes_\varphi \mathbb{R}$ is the von Neumann algebra generated by $\pi_\sigma(M) \cup \lambda(\mathbb{R}) \subset \mathcal{B}(L^2(\mathbb{R}, \mathcal{H}_\varphi))$. Note that $(\lambda(s))_{s\in\mathbb{R}}$ generates a copy of $L(\mathbb{R})$ inside $M \rtimes_\varphi \mathbb{R}$.

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- There exists a *dual weight* $\hat{\varphi}$ on $M \rtimes_\varphi \mathbb{R}$ which is a normal semifinite faithful weight on $M \rtimes_\varphi \mathbb{R}$ whose modular automorphism group $\left(\sigma_t^{\hat{\varphi}}\right)_{t\in\mathbb{R}}$ satisfies

$$\sigma_t^{\hat{\varphi}}(\pi_\sigma(x)) = \pi_\sigma(\sigma_t^\varphi(x)) \qquad \text{for all } x \in M,$$
$$\sigma_t^{\hat{\varphi}}(\lambda(s)) = \lambda(s) \qquad \text{for all } s \in \mathbb{R}.$$

- The *dual action* $(\theta_t^\varphi)_{t\in\mathbb{R}}$ on $M \rtimes_\varphi \mathbb{R}$ is given by

$$\theta_t^\varphi(\pi_\sigma(x)) = \pi_\sigma(x) \qquad \text{for all } x \in M,$$
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- Let h_φ be the unique nonsingular positive selfadjoint operator affiliated with $L(\mathbb{R}) \subset M \rtimes_\varphi \mathbb{R}$ such that $h_\varphi^{is} = \lambda_\varphi(s)$ for all $s \in \mathbb{R}$, then $\mathrm{Tr}_\varphi := \hat{\varphi}(h_\varphi^{-1}\cdot)$ defines a semifinite faithful normal trace on $M \rtimes_\varphi \mathbb{R}$ and the dual action θ^φ scales the trace:

$$\mathrm{Tr}_\varphi \circ \theta_t^\varphi = e^t \mathrm{Tr}_\varphi \qquad \text{for all } t \in \mathbb{R}.$$

  Moreover, Tr_φ is semifinite on $L(\mathbb{R})$ and is preserved by the canonical faithful normal conditional expectation $E_{L(\mathbb{R})}$ defined by $E_{L(\mathbb{R})}(\pi_\sigma(x)\lambda(s)) = \varphi(x)\lambda(s)$.

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- Hence we abstractly consider the *continuous core* $(c(M), \theta, \mathrm{Tr})$, where $c(M)$ is a von Neumann algebra with a faithful normal semifinite trace Tr and a trace-scaling action of $\mathbb{R}$, θ.
- Given a faithful normal state φ on M, there is a canonical surjective $*$-homomorphism $\Pi_\varphi\colon M \rtimes_\varphi \mathbb{R} \to c(M)$ such that

$$\Pi_\varphi \circ \theta^\varphi = \theta \circ \Pi_\varphi, \qquad \mathrm{Tr}_\varphi = \mathrm{Tr} \circ \Pi_\varphi, \qquad \Pi_\varphi(\pi_\sigma(x)) = x \ \forall x \in M.$$

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- Takesaki's duality theorem implies $c(M) \rtimes_\theta \mathbb{R} \cong M \bar{\otimes} \mathcal{B}(L^2(\mathbb{R}))$. In particular, M is amenable if and only if $c(M)$ is amenable.

- For a nonsingular action $\Gamma \curvearrowright (X, \mu)$ on a standard measure space, the *Maharam extension* $\Gamma \curvearrowright (X \times \mathbb{R}, m)$ is given by

$$g \cdot (x, t) = \left(g \cdot x, t + \log \left(\frac{d\mu \circ g^{-1}}{d\mu}(x) \right) \right),$$

and $dm = d\mu \times e^t dt$. This action is m-preserving, although m is an infinite measure.

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  and $dm = d\mu \times e^t dt$. This action is m-preserving, although m is an infinite measure.
- $c(L^\infty(X) \rtimes \Gamma) = L^\infty(X \times \mathbb{R}) \rtimes \Gamma$.
- In terms of equivalence relations, if $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$ and $L(\mathcal{R}) = L^\infty(X) \rtimes \Gamma$ and if we denote $c(\mathcal{R}) = \mathcal{R}(\Gamma \curvearrowright X \times \mathbb{R})$ then $c(L(\mathcal{R})) = L(c(\mathcal{R}))$.

Theorem 2.1

Let Γ be any group in the class $\mathcal{QH}_{reg}$ and $\Gamma \curvearrowright (X, \mu)$ any free nonsingular action on a standard measure space. Let $V \subset X$ be a non-negligible subset. Every nonamenable recurrent subequivalence relation of $\mathcal{R}(\Gamma \curvearrowright X) |_V$ is indecomposable.

Theorem 3.1

Let M be any σ-finite von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation $E_B \colon 1_B M 1_B \to B$. The following are equivalent:

(1) *There exist $n \geq 1$, a possibly nonunital normal $*$-homomorphism $\pi \colon A \to M_n(B)$ and a nonzero partial isometry $v \in M_{1,n}(1_A M 1_B)$ such that $av = v\pi(a)$ for all $a \in A$.*

(2) *There is no net of unitaries $(w_i) \subset \mathcal{U}(A)$ such that $E_B(x^* w_i y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.*

Proof of (2) $\Rightarrow$ (1).

- Fix faithful normal tracial state τ on B and define $\varphi(x) = \tau(E_B(x))$ for $x \in 1_B M 1_B$, extend to faithful normal positive functional on M.

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- Denote $\mathcal{H} := J1_B J L^2(M, \varphi)$, Mz is faithfully represented on $\mathcal{H}$ where z is central support of 1_B in M. Let e_B denote the orthogonal projection of $\mathcal{H}$ onto $L^2(B, \tau) \subset \mathcal{H}$.

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- Consider $\mathcal{N} := \mathcal{B}(\mathcal{H}) \cap (JBJ)'$, and realize that the faithful trace τ on B implies there is a canonical faithful normal semifinite trace Tr on $\mathcal{N}$ satisfying $\mathrm{Tr}(TT^*) = \tau(T^*T)$ for all bounded right B-linear maps $T\colon L^2(B, \tau) \to \mathcal{H}$. Also, $e_B \in \mathcal{N}$ with $\mathrm{Tr}(e_B) = 1$.

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- Regard Mz as a von Neumann subalgebra of $\mathcal{N}$, since it acts faithfully on $\mathcal{H}$.

Proof of (2) $\Rightarrow$ (1).

- On bounded subsets of B, strong$*$ topology coincides with $\|\cdot\|_2$-topology. So (2) implies $\exists\delta > 0$ and a finite subset $\mathcal{F} \subset 1_A M 1_B$ such that

$$\sum_{x,y\in\mathcal{F}} \|E_B(x^* w y)\|_2^2 \geq \delta \qquad \text{for all } w \in \mathcal{U}(A).$$

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- By considering $\xi = \sum_{x\in\mathcal{F}} x e_B x^* \in \mathcal{N}_+$, taking minimal element of $\|\cdot\|_{2,\mathrm{Tr}}$-norm in the weak closure of convex hull of $\{w\xi w^* \colon w \in \mathcal{U}(A)\}$, and taking a suitable spectral projection we can find a nonzero projection $p \in A' \cap 1_A \mathcal{N} 1_A$ such that $\mathrm{Tr}(p) < \infty$.

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- $p\mathcal{H}$ is a nonzero $A - B$-bimodule with finite dimension over B. Can find $\mathcal{K} \subset p\mathcal{H}$ a nonzero $A - B$-subbimodule which is finitely generate as a right B-module.

Proof of (2) $\Rightarrow$ (1).

- Let $n \geq 1$ and $q \in M_n(B)$ a nonzero projection such that there exists a right B-module isomorphism $\psi \colon \mathcal{K}_B \to (qL^2(B)^{\oplus n})_B$.

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$\square$

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- Take $\xi_i \in \mathcal{K}$ with $\psi(\xi_i) = q(0, \ldots, 0, 1_B, 0, \ldots, 0)$. Then $\xi = (\xi_i) \in M_{1,n}(\mathbb{C}) \otimes \mathcal{K}$ satisfies $a\xi = J\pi(a)^* J\xi$ for all $a \in A$.

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Definition 3.2

If either of the two equivalent conditions in Theorem 3.1 holds, then we say A *embeds into* B *inside* M and denote $A \preceq_M B$.

Lemma 3.3

Let M be a von Neumann algebra with a separable predual. Let $A \subset 1_A M 1_A$ and $B \subset M$ be unital von Neumann subalgebras with expectations. Assume that B is abelian and that for every nonzero projection $p \in A$, we have $pAp \not\preceq_M B$. Then there exists a diffuse abelian $*$-subalgebra $D \subset A$ with expectation such that $D \not\preceq_M B$.

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Proof.

- Find $z \in \mathcal{Z}(A)$ such that Az is type I and $A(1_A - z)$ has no type I direct summand.
- Fix $D_0 \subset Az$ unital maximal abelian $*$-subalgebra with expectation, it follows that $D_0 \not\prec_M B$.

Proof.

- In $A(1_A - z)$, inductively construct an increasing sequence (Q_n) of unital abelian finite dimensional $*$-subalgebras of $A(1_A - z)$ and unitaries $w_n \in Q_n$ such that $\|E_B(x_i^* w_i x_j)\|_{2,\tau\circ E_B} < n^{-1}$, where $\{x_i\}_{i\in\mathbb{N}} \subset (1_A - z)M$ is dense with respect to the $\|\cdot\|_{2,\tau\circ E_B}$-norm.

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- Set $D_1 := \bigvee_n Q_n$, then $D_1 \nprec_M B$.

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- Set $D_1 := \bigvee_n Q_n$, then $D_1 \not\preceq_M B$.
- $D = D_0 \otimes D_1$ is a unital diffuse abelian $*$-subalgebra with expectation such that $D \not\preceq_M B$.

□

Definition 3.4

An equivalence relation $\mathcal{R}$ is *amenable* if there exists a state Ω on $L^\infty(\mathcal{R})$ satisfying

$$\Omega(F) = \int_X F(x)\, d\mu(x) \qquad \text{for all } F \in L^\infty(X), \text{ and}$$
$$\Omega(u(\psi)Fu(\psi)^*) = \Omega(F) \qquad \text{for all } \psi \in [\mathcal{R}],\ F \in L^\infty(\mathcal{R})$$

Lemma 3.5

A countable pmp equivalence relation $\mathcal{R}$ is amenable if and only if for all non-negligible $\mathcal{R}$-invariant measurable subsets $U \subset X$ and all $\psi_1, \ldots, \psi_n \in [\mathcal{R}]$, we have

$$\left\| \sum_{k=1}^n u(\psi_k)1_U \otimes Ju(\psi_k)1_U J \right\|_{min} = n. \qquad (1)$$

Proof.

- By a lemma in [3], (1) is equivalent to the existence of a state Ω on $L^\infty(\mathcal{R})$ satisfying

$$\Omega(F) = \int_X F(x)\, d\mu(x) \qquad \text{for all } F \in L^\infty(X)^{\mathcal{R}}, \text{ and}$$
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$$\Omega(u(\psi)Fu(\psi)^*) = \Omega(F) \qquad \text{for all } \psi \in [\mathcal{R}],\ F \in L^\infty(\mathcal{R})$$

- So it suffices to show $\Omega(1_V) = \mu(V)$ for any $V \subset X$. Define mean $\Psi(V) := \Omega(1_V)$.

Proof.

- **Case 1:** $\mathcal{R}$ is homogeneous type I_n, $1 \leq n < \infty$.

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- **Case 1:** $\mathcal{R}$ is homogeneous type I_n, $1 \leq n < \infty$.
- $\exists V \subset X$ with $\mu(V) = \frac{1}{n}$, and $\psi_1, \ldots, \psi_n \in [\mathcal{R}]$ such that for a.e. $x \in V$, $\{\psi_1(x), \ldots, \psi_n(x)\}$ is equivalence class of x.

Proof.

- **Case 1:** $\mathcal{R}$ is homogeneous type I_n, $1 \leq n < \infty$.
- $\exists V \subset X$ with $\mu(V) = \frac{1}{n}$, and $\psi_1, \ldots, \psi_n \in [\mathcal{R}]$ such that for a.e. $x \in V$, $\{\psi_1(x), \ldots, \psi_n(x)\}$ is equivalence class of x.
- For $U \subset V$, $\psi_k(U)$ are disjoint and union is $\mathcal{R}$-invariant so:

$$n\mu(U) = \mu\left(\bigcup_{k=1}^{n} \psi_k(U)\right) = \Psi\left(\bigcup_{k=1}^{n} \psi_k(U)\right)$$
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- Then by ψ_k-invariance, $\mu(U) = \Psi(U)$ for all $U \subset \psi_k(V)$, hence for all $U \subset X$ by finite additivity.

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- Let $E\colon L^\infty(X) \to L^\infty(X)^{\mathcal{R}}$ be trace preserving conditional expectation.
- Can write $L^\infty(X)^{\mathcal{R}} = L^\infty(Y, \eta)$ with (Y, η) a standard probability space, and can write (X, μ) as a direct integral over (Y, η) of a measurable field of standard probability spaces isomorphic to $([0, 1], dx)$ (as $\mathcal{R}$ is type II_1).

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- There is an isomorphism of probability spaces $\theta\colon [0,1]\times Y \to X$ such that $F(\theta(t,y)) = F(y)$ for all $F \in L^\infty(Y) = L^\infty(X)^{\mathcal{R}}$ and a.e. $(t,y) \in [0,1]\times Y$. Also $(E(F))(y) = \int_0^1 F(\theta(t,y))\,dt$ for all $F \in L^\infty(X)$ and a.e. $y \in Y$.

Proof.

- Given $G\colon Y \to [0,1]$ define

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- Then by further partitioning $[0,1]$, one is able to show $|\mu(\mathcal{V}(G)) - \Psi(\mathcal{V}(G))| < n^{-1}$ for all n.

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$\square$

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- Fix $\epsilon > 0$ and choose F large enough, yet finite, so that $\mu(\bigcup_{n\notin F} V_n) < \epsilon$. This union is $\mathcal{R}$-invariant so same inequality holds for Ψ. Then for any measurable $U \subset X$ we have $|\mu(U) - \Psi(U)| < 2\epsilon$.

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- Denote $P_i = L(\mathcal{S}_i)$ and $e = 1_V$, then $P_1 \bar{\otimes} P_2 \subset eMe$ with expectation and hence $P_i \subset eMe$ with expectation.
- Recurrence of $\mathcal{S}_1 \Rightarrow P_1$ has no type I direct summand so condition (1) in definition implies $pP_1p \not\preceq_M L^\infty(X)$ for every nonzero projection $p \in P_1$. Then Lemma 3.3 implies $\exists A \subset P_1$ diffuse abelian $*$-subalgebra with expectation such that $A \not\preceq_M L^\infty(X)$.

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- Let $G := [c(\mathcal{S}_2) |_U]$ be the full group. For each $\psi \in G$ we have canonical unitary $u(\psi) \in pc(P_2)p$.

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- By Lemma 3.5, suffices to prove for every projection $z \in \mathcal{Z}(pc(P_2)p)$ and all $\psi_1, \ldots, \psi_n \in G$ that

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- We construct completely positive contractive maps $\varphi_i \colon \mathcal{E} \to L^\infty(X \times \mathbb{R}) \rtimes_{\mathrm{red}} \Gamma$ such that $\varphi_i(x)p \to xp$ $*$-strongly for all $x \in \mathcal{E}$.

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- For each k find partitions $(U_g^k)_{g\in\Gamma}$ and $(V_g^k)_{g\in\Gamma}$ of U such that $\psi_k(y) = g \cdot y$ for a.e. $y \in U_g^k$ and $\psi_k^{-1}(y) = g \cdot y$ for a.e. $y \in V_g^k$.

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- So $p_i := 1 - p^{1/i}$ is a sequence converging to $p = 1_U$ strongly such that $u(\psi_k)p_i, u(\psi_k)^*p_i \in L^\infty(X\times\mathbb{R})\rtimes_{\text{alg}}\Gamma$ for all $k=1,\ldots,n$ and all i.

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- Define $\varphi_i(x) = p_i x p_i$.

Proof of Theorem 2.1.

- Let $H \subset L^2_{\mathbb{R}}(Y,\nu) =: D$ be the Gaussian construction associated with the representation $\pi\colon G \to \mathcal{U}(H)$ and the proper map $c\colon G \to H \subset D$ (from the definition of $\mathcal{QH}_{\text{reg}}$) and define $\tilde{M} = (D \bar{\otimes} L^\infty(X)) \rtimes \Gamma$.

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- Define

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- Recall $(V_t)_{t\in\mathbb{R}}$ are defined on $\tilde{\mathcal{H}}$ by

$$V_t(\xi \otimes \eta \otimes \delta_h) = v_t(h)\xi \otimes \eta \otimes \delta_h,$$

where $v_t(g)(x) = \exp(itc(g)(x))$.

Proof of Theorem 2.1.

- A lemma from [2] yields that for all $x \in c(L^\infty(X)) \rtimes_{\text{red}} \Gamma$,

$$\lim_{t\to 0} \|xV_t v - V_t v x\|_\infty = \lim_{t\to 0} \|JxJV_t v - V_t vJxJ\|_\infty = 0.$$

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- Using this and condition (2) in the definition of $A \not\preceq_M L^\infty(X)$ implies $\exists \xi_t \in \tilde{\mathcal{H}} \ominus \mathcal{H}$ such that $\|\xi_t\|_{\tilde{\mathcal{H}}} \geq \delta$ for all $t > 0$ and that

$$\begin{aligned}\limsup_{t\to 0} &\|\varphi_i(u(\psi_k)z)J\varphi_i(u(\psi_k)z)J\xi_t - \xi_t\|_{\tilde{\mathcal{H}}} \\ &\leq 2\,\|(\varphi_i(u(\psi_k)z) - u(\psi_k)z)p\|_{2,\text{Tr}} \overset{i\to\infty}{\longrightarrow} 0,\end{aligned}$$

where Tr is the canonical trace on $c(M)$.

Proof of Theorem 2.1.

- Thus

$$\limsup_i \left\| \sum_{k=1}^{n} \varphi_i(u(\psi_k)z) J \varphi_i(u(\psi_k)z) J \right\|_{\mathcal{B}(\tilde{\mathcal{H}} \ominus \mathcal{H})} \geq n.$$

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- The binormal representation $a \otimes JbJ \mapsto aJbJ$ of $c(M) \otimes_{\text{alg}} Jc(M)J$ is continuous with respect to the minimal C^*-tensor norm ([1],[6]) so:

$$\limsup_i \left\| \sum_{k=1}^{n} \varphi_i(u(\psi_k)z) \otimes J \varphi_i(u(\psi_k)z) J \right\|_{\min}$$
$$\geq \limsup_i \left\| \sum_{k=1}^{n} \varphi_i(u(\psi_k)z) J \varphi_i(u(\psi_k)z) J \right\|_{\mathcal{B}(\tilde{\mathcal{H}} \ominus \mathcal{H})} \geq n$$

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- Finally, the φ_i are completely positive and contractive so

$$\left\| \sum_{k=1}^{n} u(\psi_k) z \otimes J u(\psi_k) J \right\|_{\min} \geq \limsup_i \left\| \sum_{k=1}^{n} \varphi_i(u(\psi_k) z) \otimes J \varphi_i(u(\psi_k) z) J \right\|_{\min} \geq n$$

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